Theme 2. Fundamental laws of continuum mechanics (3 hours)

Lecture 2. Laws of conservation

In classic statistical mechanics solid is considered as a system of particles which interact each other and with boundary bodies and placed in the field of external forces. The laws of the mechanics of a system of material points are assumed to be effective for such body. When being so the forces of interaction between the particles occur when they approach and disappear while they move away. The information about individual motion of the particles doesn't convey about macroscopic properties of the system. There are some methods in statistical mechanics which allow to introduce the definitions of density, velocity, internal stresses, energy, temperature, entropy and flow of heat as continuously differentiable functions of coordinates and time. In plasticity, the material is usually considered to be an homogeneous continuum. It is often also considered to be isotropic. An **homogeneous** body consists of one phase and has identical properties at all points; **isotropic** means that properties are the same in all directions.

There are some functions of Lagrange's variable called motion integral for confined system. The motion integral is called additive if it is equal to sum of motion integrals of the component parts of the system. They are: mass, momentum, angular momentum and energy. These quantities are constant. This fact allows to lay down the laws of conservation of mass, momentum, angular momentum and energy. This laws results from fundamental property of substance, motion, space and time as a form of existence of substance.

As applied to mechanics of continua these laws lead to some important conclusions. Equation of continuity follows from the law of conservation of mass, i.e. conditions of deformation of the body without void formation. Differential equations of motion follow from the law of momentum conservation. Such equations serve as a basis for a calculation of deformations and motion of continuum. The symmetry of stress tensor follows from the law of conservation of angular momentum that simplifies the dynamic equation of continuum. The law of conservation of energy is the basis of extremal energetic methods of determination of a stress-strain state.

The fundamental definition of internal stresses and strains are introduced and the relations between such quantities and temperature are postulated. This fact follows from the statistics of atom motion and interaction.

The law of mass conservation and equation of continuity

The fundamental law of Newton's mechanics is the law of mass conservation of any individual element: $m = const$, or $\frac{dm}{dt} = 0$.

Let's consider the motion of continuum in motionless system of coordinates. It is assumed that density ρ and velocity $\vec{v}$ are continuous functions of coordinates and time, i.e. $\rho = \rho(\vec{r}, t)$ or $\vec{v} = \vec{v}(\vec{r}, t)$.

Let's determine mass change per unit time in element V, which confined by closed surface S. So mass in element V is equal to $\iiint \rho\, dV$, there is change of mass inside the surface S which is equal to

$$\frac{\partial}{\partial t}\iiint \rho\, dV = \iiint \frac{\partial \rho}{\partial t} dV . \qquad (2.1)$$

This change of mass is equal to flow of mass throw the surface S per unit of time which is equal to $\oiint \rho \vec{v}\vec{n} dS$. Using the formula of Ostrogradky-Gauss one can obtain:

$$\frac{\partial \rho}{\partial t} + div \rho \vec{v} = 0 . \qquad \mathbf{(2.2)}$$

It is one of the basic equation in mechanics of continua which is called equation of continuity in Eiler's variables. It may be changed by applying functional of material differentiation to the function of density $\rho(\vec{r}, t)$:

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + v_k \frac{\partial \rho}{\partial X_k} .$$

Then:

$$\frac{\partial \rho}{\partial t} = -\rho\, div \vec{v} \qquad (2.3)$$

This relationship imposes restrictions on a velocity of the points of continuum and may be used under big displacements of continuum. The constancy of density follows from the fact that any individual element remains permanent during motion ($\rho = const$). Therefore for incompressible continuum $\frac{d\rho}{dt} = 0$ and from equation (2.3) one obtains the equation of continuity:

$$div \vec{v} = 0 . \qquad (2.4)$$

or

$$\frac{\partial v_1}{\partial X_1} + \frac{\partial v_2}{\partial X_2} + \frac{\partial v_3}{\partial X_3} = 0 . \qquad (2.5)$$

The equation of motion and equilibrium

For the material particle moving under the influence of the sum of the forces $\vec{F}$ the second Newton's law can be expressed in the form of momentum equation:

$$\frac{d}{dt} m\vec{v} = \vec{F} \qquad (2.6)$$

Generalizing momentum equation in case of individual element of continuum V and projecting on the axes x_1, x_2, x_3, one obtains:

$$\frac{d}{dt}\iiint \rho \vec{v}_i dV = \iiint \rho \vec{f}_i dV + \oiint \vec{\sigma}_{ni} dS \qquad (2.7)$$

Let's determine differential form of the equation of motion. Using expression for symmetric stress tensor and applying the Ostrogradsky-Gauss equation one obtains:

$$\iiint \left(\frac{\partial \sigma_{ij}}{\partial x_j} + \rho f_i\right) dV = \frac{d}{dt}\iiint \rho v_i dV . \qquad (2.8)$$

or

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho f_i = \rho \frac{dv_i}{dt} dV . \qquad (2.9)$$

Expression (2.9) is called equation of motion and it is the basic differential equation of continuum mechanics. It is valid for any continuum: solid, liquid, gaseous one. All quantities in Eq. (2.9) are the functions of coordinates x_1, x_2, x_3 and time t, i.e. they are represented in Eiler's variables. Partial derivative $\frac{\partial v_i}{\partial t}$ can be determined according to the functional of material differentiation.

In case of equilibrium ($\frac{\partial v_i}{\partial t}$=0) the inertial terms may be neglected ($\rho \frac{\partial v_i}{\partial t}$=0), so the equation of motion transforms in equilibrium equation:

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho f_i = 0.$$

If the gravitational forces may be neglected one can obtain:

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0. \qquad (2.10)$$

Multiplying vectorly Eq. (2.6) by radius vector of particle under consideration one obtains the angular momentum equation:

$$\frac{d}{dt}(\vec{r} \times m\vec{v}) = \vec{r} \times \vec{F},$$

which is the corollary of the second Newton's law. Vector quantity is called angular momentum of particle motion.

The angular momentum equation can be expressed for the system of material particle with the mass equal to m_i which is moving at the velocity v_i and can be generalized for an individual element of continuum V confined with the surface S:

$$\frac{d}{dt}\iiint_V (\vec{r} \times \rho\vec{v})dV = \iiint_V (\vec{r} \times \rho\vec{f})dV + \oiint_S (\vec{r} \times \vec{\sigma}_n)dS.$$

The symmetry of the stress tensor is the consequence of the angular momentum equation in case distributed force couples are absent.

The theorem of the live forces is one of the most important corollary of the equation of continuum motion. The definition of a work is introduced as degree of the force action on a material particle under its movement. Here the work determines force action on a change of the velocity module of the moving particle.

Elementary work of the external surface forces $dA^{(e)}_{sur}$:

$$dA^{(e)}_{sur} = \oiint_S \vec{\sigma}_{ni} v_i dtdS = \oiint_S \sigma_{ij} n_j v_i dtdS. \qquad (2.11)$$

Elementary work $dA^{(e)}{}_m$ of the external gravitational forces $\rho\vec{f}dv$ under infinitesimal displacement $d\vec{r} = \vec{v}dt$ can be determined by summing over space V:

$$dA^{(e)}_m = \iiint_V \rho\vec{f}d\vec{r}dV = \iiint_V \rho\vec{f}\vec{v}dtdV = \iiint_V \rho f_i v_i dtdV. \qquad (2.12)$$

The work of internal surface stresses is:

$$dA^{(i)}_{sur} = \iiint_V \sigma_{ij}\xi_{ij} dtdV,$$

where $\xi_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$ is the strain rate tensor.

The change of kinetic energy or its differential can be determined from the equation of motion as:

$$dE_m = \iiint_V \rho v_i dv_i dV = \iiint_V v_i \frac{\partial \sigma_{ij}}{\partial x_j} dtdV + \iiint_V v_i \rho f_i dtdV. \qquad (2.13)$$

So the theorem of a live forces can be expressed as:

$$dE_m = dA^{(e)}_{sur} + dA^{(i)}_{sur} + dA^{(e)}_m. \qquad (2.14)$$

Thus in case of continuum motion the kinetic energy differential of an individual element is equal to the sum of elementary works of external, internal surface forces and external gravitational forces which act on that element.

The relationship (2.14) by nature is energetic, but it isn't the energy conservation law, because it doesn't take into account other kind of energy. The theorem of the live forces can be interpreted as the law of conservation when there isn't transformation of the mechanical energy into the heat one.

Inasmuch as differential of the internal energy dU_m under adiabatic process is equal to elementary work of internal surface forces with opposite sign, the theorem (2.14) can be represented as

$$\frac{dE_m}{dt} + \frac{dU_m}{dt} = N^{(e)}_{sur} + N^{(e)}_m, \qquad (2.15)$$

where $N^{(e)}_{sur} = \oiint \sigma_{ij} n_j v_i dS$, $N^{(e)}_m = \iiint \rho f_i v_i dV$ – are the powers of external surface and gravitational forces.

The equation of the mechanical energy establishes the relation between the change of the mechanical energy of a chosen element of continuum and the power of the external and gravitational forces.

Heat balance equation. The first law of thermodynamics

In industrial metalworking processes most of deformation energy is transformed into thermal one, which manifests itself as heat generation. This heat generation causes the temperature to increase within the workpiece and results in a varying thermomechanical behavior of the material. In an inhomogeneous deformation process, the external friction losses raise the temperature at the die-workpiece interface also. Therefore, the temperature influence on the behavior of material under loading must be taken into account.

The thermodynamics deals with the laws of energy transformation. One of the basic conception of the thermodynamics is a system state and state variables.

The definition of a thermodynamic system determines the aggregate of material bodies interacting with each other and with surroundings. In order to determine certain physical conditions the thermodynamic parameters must be introduced. They represent substance properties which do not depend on a quantity of this substance. Parameter may be internal and external. The system approaches to the state called thermodynamic equilibrium. If there isn't equilibrium and there are gradients of state variables, such state is called nonequilibrium.

There must be dependence between the state variables called equation of state: $f(\mu_1, \mu_2, ..., \mu_3)$.

Two kind of processes are possible between two states of system. If after accomplishment of process the system returns forward and backward to its initial state, such process is a reversible one. Irreversible process is impossible without any changes. Such processes lead to the dissipation of an energy.

Deformed body under conditions of plastic working may be considered as non-isolated thermodynamic system where nonequilibrium, irreversible thermodynamic processes take place. The heat and mechanical energy inflow is of great importance in continuum mechanics.

According to the first law of thermodynamics there is a single-valued function $E=E(\mu_1, \mu_2, ..., \mu_3)$, called full energy of system, exact differential of which is equal to the sum of elementary works of external surface and gravitational forces $dA^{(e)}$, elementary inflow of heat $dQ^{(e)}$ and elementary inflow of other kinds of energy dQ^*:

$$dE = dA^{(e)} + dQ^{(e)} + dQ^*, \qquad (2.16)$$

Full energy variation in system takes place due to change of its kinetic energy dE_m and internal energy dU_m:

$$dE = dE_m + dU_m. \qquad (2.17)$$

The first law of thermodynamics can be expressed as:

$$dE_m + dU_m = dA^{(e)} + dQ^{(e)} + dQ^*, \qquad (2.18)$$

For the adiabatic processes, where all the work done on an element is converted into heat and none is lost $dQ^{(e)} = 0$ and if $dQ^* = 0$ the Eq. (2.18) is transformed into the law of a conservation of mechanical energy (2.14). Taking away Equ. (2.14) from (2.18) one can obtain:

$$dU_m = -dA^{(i)}_{sur} + dQ^{(e)} + dQ^*. \qquad (2.19)$$

which is called heat balance equation and can substitute the law of conservation of energy (2.18).

If the elementary inflows of external energy to unit of mass $dq^{(e)}$ and $dq^{(*)}$ are introduced one can obtain heat balance equation for an infinitesimal particle of continuum:

$$dU = \frac{1}{\rho}\sigma_{ij}\xi_{ij}dt + dq^{(e)} + dq^{*}, \tag{2.20}$$

where U is a internal energy density, $\xi_{ij} = \xi_{ji} = \frac{1}{2}(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_J}{\partial x_i})$.

The equation (2.20) is a universal one used for various processes. It is a connecting link between the continuum mechanics and thermodynamics. If it is impossible to neglect thermal effects, the law of conservation of mechanical energy (2.14) shall be replaced by the first law of thermodynamic (2.18), in addition the equation of thermal conductivity should be included.

Equation of thermal conductivity

Any process of heat transfer is called heat exchange. The heat can be transferred by means of one of the three mechanisms: thermal conductivity, convection and radiation. The convection is of the greatest importance in continuum mechanics.

Inasmuch as it is impossible to solve curtain problems of continuum mechanics using Eq.(2.19), one often uses some additional assumptions idealising process. They are: (1) adiabatic process ($dQ^{(e)} = 0$) which takes place when one deals with heat-insulated and quickly passed processes and (2) isothermal process ($dT/dt=0$) which takes place when temperature can be considered to be constant or it is function of the time only.

From the heat balance equation one can obtain equation of thermal conductivity:

$$c\rho\frac{dT}{dt} - \sigma_{ij}\xi_{ij} - div(\lambda\ grad\ T) = 0, \tag{2.21}$$

where $c = \frac{dq^{(e)}}{dT}$ is the mass heat capacity, λ is the heat conductivity coefficient.

Such solution contains constants of integration, therefore it is ambiguous. Because of this fact it is necessary to formulate boundary value problem. It may include initial conditions (temperature distribution when t=0) and boundary conditions which are given on the surfaces confining continuum under consideration.

The second law of thermodynamics

The first law of thermodynamics doesn't determine the direction of an energy transformation so both transformation of the mechanical energy into the heat and reverse process are equally possible.

All processes are irreversible and have a tendency to establish an equilibrium of a system. The second law of thermodynamics postulates that it is impossible to obtain energy at the expense of bodies being in thermodynamic equilibrium.

One may say that under reversible processes $\frac{Q}{T}$ is absorbed as much as it is released. Such a quantity is called entropy. They say that variation of entropy is zero for any reversible cycle. For any cycle:

$$\oint_L \frac{dQ^{(e)}}{T} \le 0, \tag{2.22}$$

where $\frac{dQ^{(e)}}{T} = dS$ – variation of entropy, i.e. $TdS = dQ^{(e)} + dQ'$, where dQ' – noncompensated heat. For mass unit:

$$Tds = dq^{(e)} + dq'; \qquad dq' = \frac{dQ'}{dm} \ge 0. \tag{2.23}$$

For isolated system the entropy value is maximum when system is in thermodynamic equilibrium.

$$s = k\ ln W, \tag{2.24}$$

where k is a Bolcsman's constant, W is a measure of probability of state.

One can combines the second law of thermodynamics and heat balance equation:

$$TdS \ge dU_m + dA^{(i)}_{sur} + dQ^{*}. \tag{2.25}$$

In addition to the heat balance equation the equation of state is a connecting link between continuum mechanics and thermodynamics. During the deformation of continuum with internal friction (viscoelastic and viscoplastic) the entropy occurs. It is very phenomena leads to irreversible process.

Fundamental equations of the mechanics of continua

There are five independent equations based on the laws of conservation. They are: (1) equation of continuity, (2, 3, 4) equations of motion and (5) heat balance equation. This system can be complemented by the second law of thermodynamics. Finally we have the system of fundamental equations of the mechanics of continua:

$$\begin{cases} \frac{d\rho}{dt} + \rho\ div\ v = 0; \\ \frac{\partial\sigma_{ij}}{\partial x_j} + \rho f_i = \rho\frac{dv_i}{dt}; \\ dU = \frac{1}{\rho}\sigma_{ij}\xi_{ij}dt + dq^{(e)} + dq; \\ Tds = dq^{(e)} + dq', \quad dq' \ge 0. \end{cases} \tag{2.26}$$

If it is possible to neglect the dissipation of energy, the system of fundamental equations can be expressed as:

$$\begin{cases} \frac{d\rho}{dt} + \rho\ div\ \vec{v} = 0; \\ \frac{\partial\sigma_{ij}}{\partial x_j} + \rho f_i = \rho\frac{dv_i}{dt} \end{cases} \tag{2.27}$$

Such system of four differential equations in partial derivatives has ten unknown functions of coordinate and time. The unknown are: density ρ, three component of velocity v_i (or displacement u_i) and six unknown components of the stress tensor σ_{ij}. In addition boundary conditions are prescribed.

To close the system it is necessary another six equations containing only dynamic and kinetic parameters, which often represent strain-stress relationships. Such relationships are called mechanical equations of state. Besides the temperature field as a rule is considered to be known or the equation of thermal conductivity is resolved separately from mechanical problem. In case of isothermal processes the temperature is considered to be constant and the problem is mechanical one.

In common case when the mechanical and thermal phenomena are interdependent it is necessary to use the entire system of equations. Supposing that gravitation forces f_i and elementary inflows of an energy dq^{*} are defined, the unknown functions are: ten quantities ρ, v_i, σ_{ij}, three components of the vector of the thermal flow density q_i, density of the internal energy U, density of the entropy s and absolute temperature.

So it is necessary to find another ten equations to determine the system. They are: six equations of state, three equations determining the law of thermal conductivity and thermodynamics equation of state. By means of equations of state they introduce some idealised continuum because of complicated behaviour of materials. It is necessary to set initial and boundary conditions to solve the system of differential equations.